





85  
1006



# BEER

**FACULTY WORKING  
PAPER NO. 1006**

THE LIBRARY OF THE

**FEB 21 1984**

UNIVERSITY OF ILLINOIS  
AT URBANA-CHAMPAIGN

**An Analysis of Nonlinearities in Asset Pricing:  
Methods and Implications**

*Bill McDonald  
Cheng F. Lee*

College of Commerce and Business Administration  
Bureau of Economic and Business Research  
University of Illinois, Urbana-Champaign



# BEBR

FACULTY WORKING PAPER NO. 1006

College of Commerce and Business Administration

University of Illinois at Urbana-Champaign

January 1984

An Analysis of Nonlinearities in Asset Pricing:  
Methods and Implications

Bill McDonald  
College of Business Administration  
University of Notre Dame

Cheng F. Lee, Professor  
Department of Finance

University of Notre Dame and University of Illinois, respectively.  
Data for the study were provided by the Jesse H. Jones Research  
Data Base.

Digitized by the Internet Archive  
in 2011 with funding from  
University of Illinois Urbana-Champaign

<http://www.archive.org/details/analysisofnonlin1006mcdo>

# AN ANALYSIS OF NONLINEARITIES IN ASSET PRICING:

## Methods and Implications

### ABSTRACT

Using a generalized specification of the single-factor market model, this study examines the sources of statistical anomalies previously found in estimating the market model. Two generalized models are sequentially developed for juxtaposition with the more common linear and logarithmic specifications. The extended models include a fully generalized functional form specification and a heteroscedastic process. The series of models allows the diverse remedial effects of the functional form transformation to be isolated for more detailed examination. The results indicate that previous findings of significant "nonlinearities" are primarily attributable to nonnormalities and unequal variance. The hypothesis of linearity in the market model relationship that evolves from the various asset pricing theories is supported for both the linear and logarithmic specifications. The effects of the heteroscedastic specification in reducing the level of kurtosis observed in the market model residuals provide additional support for the subordinated normal hypothesis.

### I. INTRODUCTION

From the variety of asset pricing theories subsumed in the state-preference framework of Arrow (1964) and Debreu (1959), the Capital Asset Pricing Model (CAPM) and the Arbitrage Pricing Theory (APT) emerge as the predominant paradigms in recent Finance literature. Although the CAPM and APT evolve from distinctly different theoretical foundations, they both provide for a return generating process as specified in the single-factor market model, expressed for a given security as

$$R_t = \alpha + \beta R_{mt} + \varepsilon_t, \quad (1)$$

where:

$R_t$  = the return in time  $t$  for the security,

$R_{mt}$  = the value-weighted market return in time  $t$ ,

$\alpha, \beta$  = estimated parameters, and

$\varepsilon_t \sim N(0, \sigma^2)$ .

Using functional form techniques, a number of studies (e.g., Lee 1976b, Lee 1977, and Fabozzi, Francis, and Lee 1980) document the presence of significant transformations in the market model relationship. In some cases, the authors go on to label their findings as evidence of significant "nonlinearities" in the returns model. Although this conclusion is statistically valid, it has more important implications in the context of asset pricing theories. The linear relationship between risk and return is a critical result of both the CAPM and APT specifications. Nonlinearities reported in functional form applications can be attributed to three possible sources: 1) nonnormalities in the relationship, 2) a heteroscedastic error variance, or 3) a nonlinear (nonadditive) relationship between the variables. Thus, it is important to determine the actual source of significant transformations in the market model by partitioning these effects. As an artifact of examining the linearity assumption, information on other statistical anomalies in the market model is also provided. Basically, this study uses a series of specifications including generalized functional forms and a heteroscedastic process to address distributional and linearity issues that have surfaced in previous research.

The econometric models to be tested are derived sequentially in Section II. References to previous studies and theoretical considerations will be incorporated in the model development. The third section details the estimation procedure and data base. Section IV of the paper presents the empirical results. The final section presents a compendium of the pertinent findings.

## II. MODEL DEVELOPMENT

### A. The Linear Specification

The ubiquitous linear specification of the market model must obviously be included as a form to be compared with more general statistical and



theoretical formulations. The single-factor market model in its linear form, as specified in equation (1), appears in innumerable studies, with its popularized "beta" permeating virtually all of the business disciplines. Within the purview of Finance, the model plays an important role in testing the CAPM, capital budgeting, abnormal performance and market efficiency. Concurrent econometric studies of the model indicate a need for a more detailed examination of the conformity of the empirical model to the usual regression assumptions. The discovery of these statistical aberrations suggests a more general econometric form to be tested. More importantly, many of the deviations from the simple linear model have significant theoretical implications for the underlying return generating process.

#### B. The Logarithmic Model

The logarithmic form of the market model appears as

$$R'_t = \alpha + \beta R'_{mt} + \epsilon_t, \quad (2)$$

where both the return of the individual securities and the market return are transformed as  $R' = \ln(1+R)$ . The logarithmic model is frequently proposed as a possible solution to the investment horizon problem inherent in the linear specification, and also as a remedy to deviations from normality observed in security returns. Mandelbrot (1963) indicates that the log return measure is frequently used to reduce unequal variances in cases where the standard deviation of return is proportional to price. Rosenberg and Marathe (1979) provide evidence that the log model also reduces the skewness observable in market model residuals.

Beyond the statistical remedies provided by the log model, the specification also implies an alternative theoretical form. In the linear specification the returns used to test the model are assumed to be single period returns measured over the appropriate investment horizon. Since the linear return measure is not invariant to compounding, if the observed return measure (usually one month) does not correspond to the investment horizon, the estimated parameters will be dependent on the time interval. As detailed in Rosenberg and Marathe (1979), the log return model is preserved under compounding and the estimates of beta appear to be invariant to the observed horizon. The disadvantage of the log specification is that it does not allow for the aggregation of returns in the requisite form of the CAPM, although Lintner (1975) provides evidence that this flaw is not crucial.

The selection of a linear versus a logarithmic specification has not been thoroughly assayed. Although the linear-log issue is not the predominant concern of this study, the nature of the subsequent models to be discussed allows this issue to be examined in more detail. The logarithmic model will provide another benchmark of comparison for the more general specifications. Unfortunately, a simple direct comparison of the linear and logarithmic models is not possible, since neither corresponds to a restricted form in relation to the other.

### C. Generalized Functional Form

The functional form of the market model can be generalized by applying the transformation of Box and Cox (1964), as

$$(1+R_t)^{(\lambda_1)} = \alpha + \beta(1+R_{mt})^{(\lambda_2)} + \varepsilon_t, \quad (3)$$

where:

$$x^{(\lambda)} = \begin{cases} (x^\lambda - 1)/\lambda & \lambda \neq 0 \\ \ln x & \lambda = 0. \end{cases} \quad (4)$$

Applying the transformation in this manner, the model reduces to the linear specification when  $\lambda_1 = \lambda_2 = 1$  and the logarithmic specification when  $\lambda_1 = \lambda_2 = 0$ .

The parameters of the model are estimated using maximum likelihood techniques with the log-likelihood function given by

$$\ln L = -T \ln \sigma + 1/2 \frac{\sum_t \epsilon_t^2}{\sigma^2} + J, \quad (5)$$

where  $J$  is the Jacobian of the inverse transformation of  $R_t^{(\lambda_1)}$  to  $R_t$  given by

$$(\lambda_1 - 1) \sum_t \ln R_t. \quad (6)$$

In some applications, the family of power transformations defined in equation (4) is appended to a model simply to provide for better estimates of the standard regression parameters by allowing the data to specify the appropriate functional form (see, for example, Lee 1976a). In other applications, the estimated value of the transformation parameter plays an important role in the interpretation of the underlying theoretical model. For example, various applications of the power transformation have appeared in estimates of the supply/demand for items where the elasticities are dependent on the estimated transformation parameter (e.g., Spitzer 1976 or Chang 1977). In empirical tests of production functions, the transformation parameter provides an estimate of the elasticity of substitution (see Ramsey and Zarembka 1971).

Because of the econometric issues addressed by the transformation, application of the generalized functional form (GFF) methodology to the market model has a number of important theoretical implications. The transformation, in a limited form, has been applied in other studies of the market model, but not in a manner that fully utilizes the wealth of information provided in its estimation.

The transformation defined in equation (4) assumes that for some value of  $\lambda$  the transformed observations will be normally distributed with constant variance, and expectation given by an appropriate linear model.<sup>1</sup> Thus, the transformation seeks to correct for nonlinearities, unequal variance, and non-normalities. Each of these characteristics has a crucial role in the theoretical and econometric application of the market model. Unfortunately, in casual application of the GFF method, these aberrations are corrected simultaneously and provide no specific interpretation of the estimated transformation. Although no exact method has been proposed to isolate the various remedial effects, the method developed in this study attempts to disaggregate the effects using a series of tests.

### Nonlinearities

In a previous study, Lee (1976b) concludes that there are significant "nonlinearities" (approximately  $1/3$  of the securities sampled) in estimating the market model. Care must be taken in using the term nonlinear. Although the significance of the Box-Cox transformation makes the model intrinsically nonlinear, it has not been determined whether this transformation is in fact attributable to nonlinear effects or to the other statistical irregularities

---

<sup>1</sup>Discussion of the transformation parameter is, for convenience, in terms of a single  $\lambda$ . The development can readily be extended to the multiple-transformation format of equation (3).

previously discussed. The conclusion of a linear relationship in the context of the market model (although derived from distinctly different foundations), is common to both the CAPM and APT. A critical test in establishing the empirical foundation for asset pricing theory is therefore a test of linearity. This can be addressed only if the confounding effects occurring in the estimation of  $\lambda$  can be effectively eliminated.

### Nonnormalities

As previously noted, another factor influencing the estimate of  $\lambda$  is the transformation's attempt to make the error terms more nearly normal. Fama (1965), and others, have established that security returns tend to be positively skewed and leptokurtic (the latter problem will be subsequently discussed). Although Draper and Cox (1969) show that the estimate of  $\lambda$  is relatively robust to minor deviations from normalacy, the presence of nonnormalities is addressed by monitoring distributional measures of the error terms in the empirical models. The heteroscedastic model proposed in the following section, besides addressing unequal variances, also could contribute to eliminating certain nonnormalities occurring in the observed returns.

The effects of nonnormalities in the GFF model can be expressed explicitly by first noting that the condition for consistent estimation of  $\lambda$  in the log-likelihood ( $\ln L$ ) of equation (5) is, asymptotically,

$$E \left[ \frac{\partial \ln L}{\partial \lambda'} \right]_{\lambda' = \lambda} = 0. \quad (7)$$

Zarembka (1974) examines these effects for the case of a single transformation ( $\lambda$ ) of the dependent variable. Selecting units of measure so that  $E[\Sigma \ln(1+R_t^{(\lambda)})] = 0$  and letting  $\theta_t$ ,  $\gamma_t$ , and  $\kappa_t$  equal the coefficient of

variation for  $R_t^{(\lambda)}$ , the skewness of  $R_t^{(\lambda)}$  and the kurtosis of  $R_t^{(\lambda)}$ , respectively, provides

$$E \left| \frac{\partial \ln L}{\partial \lambda'} \right|_{\lambda'=\lambda} = \frac{1}{\lambda \sigma^2} \sum_t \sigma_t^2 [\theta_t (1/2 \gamma_t^{-1/6} \theta_t \kappa_t) + \lambda E(\ln R_t)]. \quad (8)$$

In the case of nonnormal error terms that are identically distributed and homoscedastic,  $\sigma_t^2 = \sigma^2$ ,  $\gamma_t = \gamma$ ,  $\kappa_t = \kappa$ , and

$$E \left| \frac{\partial \ln L}{\partial \lambda'} \right|_{\lambda'=\lambda} = -1/\lambda \sum_t \theta_t (1/2 \gamma^{-1/6} \theta_t \kappa_t). \quad (9)$$

The first order conditions in  $\theta_t$  imply that  $\lambda$  is consistently estimated when  $\gamma=0$ . Under reasonable conditions the second order condition is approximately  $\gamma=(1/3)<\theta$ . These results indicate that if  $\kappa$  is not large and the error distribution is reasonably symmetric, then the estimate of  $\lambda$  is consistent.

### Heteroscedasticity

Also included in the remedial realm of the GFF transformation is its attempt to correct for nonconstant variance. If the objective of interpretation in estimating  $\lambda$  is to determine under what conditions the model is linear, then the affect of nonnormalities and nonconstant variance can be considered as a bias.

Zarembka (1974) shows that the estimate of  $\lambda$  is biased (in the sense previously described) in the presence of heteroscedasticity. Under conditions of normality and nonconstant variance, equation (9) is approximately equal to

$$E \left| \frac{\partial \ln L}{\partial \lambda'} \right|_{\lambda'=\lambda} = -(1/\sigma^2) \sum_t \frac{\text{var}(R_t)}{[E(R_t)]^{2(1-\lambda)}}. \quad (10)$$

For consistent estimates of  $\lambda$ , when  $\lambda=1$  the variance must be constant, as  $\lambda$  approaches zero the coefficient of variation must be constant. More generally, the estimate of  $\lambda$  is biased in the direction of stabilizing the error variance. The bias can be negative or positive depending on the relationships in equation (10). In the next section, the GFF market model is reformulated in an attempt to correct for heteroscedasticity and therefore eliminate the bias occurring in the estimation of  $\lambda$  attributable to nonconstant variance.

#### D. The Heteroscedastic-GFF Model (HET-GFF)

Substantial evidence of heteroscedasticity in the market model is provided by a number of studies (e.g., Bey and Pinches 1980 or McDonald and Morris 1983). From the results of previous studies, the occurrence of heteroscedasticity appears to be a function of the time period studied. For a time interval overlapping the one used in this study, Bey and Pinches find significant heteroscedasticity in approximately 45 percent of the securities sampled. Using a basic heteroscedastic specification, Lahairi and Egy (1981) test a model that provides for the joint estimation of functional form under conditions of unequal variance. Their results confirm the bias in estimating  $\lambda$  in the presence of heteroscedasticity and emphasize the importance of simultaneous testing of functional form and heteroscedasticity. (i.e., in isolated tests, a significant functional form transformation could be attributable to heteroscedasticity. Similarly, significant heteroscedasticity could be a result of nonlinearities of the model specification.) A similar reformulation will be applied to the GFF model of equation (5). Consistent with previous studies, the variance is assumed to be a function of  $R_{mt}$  and is specified as

$$\sigma^2 = a + bR_{mt} + cR_{mt}^2 . \quad (11)$$

The selection of this particular variance structure is based on the simulation results of Goldfeld and Quandt (1972). Using this variance structure in a maximum likelihood model, they find that for a sample size of 90 the model is consistently superior in correctly identifying heteroscedasticity. This specification is also shown to provide a level of efficiency surprisingly close to the unattainable GLS model. In tests of overspecification (i.e.,  $c=0$ ) and underspecification, Goldfeld and Quandt conclude that it is best to be overly generous in specifying the heteroscedastic process.

The GFF model of equation (5) can be extended to a heteroscedastic specification using the variance structure of equation (11) as

$$\ln L = \sum_t -\ln(a + bR_{mt} + cR_{mt}^2)^{.5} - \frac{(1/2) [R_t^{(\lambda_1)} - (\alpha + \beta R_{mt}^{(\lambda_2)})]^2}{a + bR_{mt} + cR_{mt}^2} + (\lambda_1 - 1) \sum_t \ln R_t. \quad (12)$$

As previously noted, the estimated functional form parameter is affected by heteroscedasticity. A random coefficients model is indistinguishable from a heteroscedastic model where  $\sigma^2 = \sigma_0^2 + \sigma_1^2 R_{mt}^2$  (see McDonald 1983). Thus the presence of a random coefficient will bias the functional form estimates unless accounted for in the estimation method.<sup>2</sup> The maximum likelihood model of equation (12) and the corresponding variance structure is sufficiently rich to include a random coefficient process, and in addition, a model that is heteroscedastic with respect to the square of the expected value of the dependent variable, where  $\sigma^2 = [E(R_t)]^2$ .

---

<sup>2</sup>Sunder (1980) provides evidence on the presence of random coefficients in estimating the market model relationship.



The purpose of extending the GFF model to a heteroscedastic specification, beyond the statistical interest of increased efficiency, is twofold. First, and most apparent from the previous section, the model is an attempt to eliminate the affects of heteroscedasticity on the estimation of the transformation parameters of the GFF model.

A second and equally important function of the heteroscedastic model is to address the distributional deviations from normality occurring in the return generating process. Noting the persistence of high levels of kurtosis in security returns, two theories have been posited to explain this phenomenon. Mandelbrot (1963) first popularized the theory that security returns are generated from a more general family of distributions--the stable Paretian. An equally valid proposition is that the returns are generated from a subordinated stochastic process consisting of a mixture of normal distributions. This latter theory suggests that the observed returns all emanate from an underlying normal distribution; however at each value the variance of the distribution may not be identical. If in fact this describes the underlying return generating process, then we would expect to observe an aggregate distribution with high levels of kurtosis. Clark (1973) provides a thorough development of the distribution theory underlying the subordinated model. Both Westerfield (1977) and Clark provide limited empirical evidence that the subordinate hypothesis is a more appropriate descriptor of security returns than the stable Paretian.<sup>3</sup>

The subordinate hypothesis can be addressed by examining the residuals of the series of proposed models. If the subordinate model is appropriate, the HET-GFF specification should also serve to reduce the kurtosis introduced into

---

<sup>3</sup>It should be noted, however, that both studies assume the heteroscedasticity to be a function of trading volume.

the market model through security returns. Furthermore, the extended model should serve to isolate the remedial effects of the transformation parameters by adjusting for unequal variances.

#### E. Synopsis of Proposed Models

A variety of theoretical and empirical issues can be assayed using the series of models previously described. The linear and logarithmic market model specifications, the predominant econometric methodologies appearing in Finance literature, provide a benchmark for comparison with the fully generalized model. The final model is an eclectic form that incorporates the issues of functional form and unequal variance.

Although previous studies have applied the GFF methodology, none have attempted to isolate the econometric effects so that the estimated transformation can be explicitly interpreted. The majority of previous studies applying the transformation have also restricted the GFF model by assuming a single transformation across all variables (i.e.,  $\lambda_1 = \lambda_2$ ). The more general specification, where the security returns and market returns have distinct transformation parameters, has a number of important implications. Most importantly, note that the transformation of an independent variable should concentrate on the linearity issue, whereas the dependent variable is random, and the issues of normality and constant variance occlude the ability to interpret the transformation. Thus separate transformations should concentrate the biases in estimating the transformations on  $\lambda_1$ .

The series of proposed models allows indirect comparison of the most frequently assumed linear and logarithmic models to a more general specification. Incorporating a heteroscedastic process in the GFF model allows the error distributions of the various specifications to be compared in order to identify sources of nonnormalities. If the effects of heteroscedasticity and

nonnormalities can be isolated, it then becomes possible to test the linearity of the market model as posited by predominant asset pricing theories.

### III. ESTIMATION AND DATA

The parameters for the GFF model of equation (5) and its extension to the heteroscedastic specification of equation (12) are estimated using full-information maximum likelihood techniques. Some concern arises given the presence of nonnormalities in estimating a function that assumes normality. Zarembka (1974) indicates that the generalized model is robust with respect to "reasonable" nonnormalities. Subsequent results indicate that although non-normalities are present, they are minimal in the more general specification. In estimating the generalized models, the functions were all well-behaved and converged quite rapidly, given the number of parameters being considered. The conformance of the empirical models to the underlying assumptions appears to be sufficient to avoid any critical misspecifications.

Data for the study were taken from the CRSP tapes for the period January, 1974 to December, 1980. A total of 1,042 securities had complete information for the 84 month time interval. The market return was measured using the value-weighted index reported on the CRSP tapes. A more extensive sequence of time-series tests was not possible because of the computational burden associated with the empirical models. The generally accepted time interval for estimating betas is from four to nine years. A seven-year interval was selected to provide an empirical model that would avoid stability problems yet be of sufficient size to attain the asymptotic properties underlying the estimation techniques.

The four empirical specifications--linear, logarithmic, GFF, and HET-GFF--are estimated for each of the 1,042 securities. This makes

presentation of the results somewhat cumbersome; therefore, descriptive statistics are used to summarize the parameter estimates across all securities.

#### IV. RESULTS

##### A. Comparison of the Model Specifications

The importance of extending the linear and logarithmic models to the more general specifications is examined using the likelihood ratio test. Under general conditions,  $-2(\ln L_1 - \ln L_2)$  is distributed as a  $\chi^2_{(r)}$ , where  $L_1$  is the likelihood of the constrained model,  $L_2$  is the unconstrained likelihood value, and  $r$  is the number of parameters for which  $L_1$  specifies given values. A direct comparison of the linear and logarithmic model is not possible since neither specification corresponds to a restricted form in relation to the other. The percentage of securities where one form provides a significant improvement over a corresponding restricted form is shown in Table 1. (All statistical tests in the study will be at the  $\alpha = .05$  level unless otherwise stated.)

---

Insert Table 1 about here

---

In comparison to the linear market model, the GFF model is superior in more than half the cases. Whether this is attributable to corrections for nonnormality, heteroscedasticity, or true nonlinearities is not determinable at this point. The logarithmic model does not appear to resolve this problem. However, it does have substantially fewer cases where the GFF extension is necessary. A similar conclusion can be made in comparing the HET-GFF model to the linear and logarithmic forms.

The HET-GFF specification provides an improvement beyond the GFF model in approximately 39 percent of the cases. Thus the more general forms merit

Table 1

Specification comparisons:  
 Percentage of securities with significant  
 difference between the general and restricted  
 market model forms<sup>a</sup>

Generalized Form	<u>Restricted Counterpart</u>		
	LIN	LOG	GFF
GFF	54.8%	18.9%	---
HET-GFF	67.6%	38.8%	38.8%

<sup>a</sup>Tests are performed for each of the 1,042 securities using the Likelihood Ratio Method with  $\alpha = .05$ .

additional attention given the observed frequency with which they contribute significant information.

### 8. Heteroscedasticity and the Market Model

The purpose of including a heteroscedastic process in the model specifications is to determine the impact of unequal variances on the distribution of the market model residuals and the effect of unequal variances on estimates of the transformations. Testing for the presence of heteroscedasticity is not a predominant purpose of this study and can be found elsewhere in previous literature. However, evidence related to this issue is an artifact of the estimation process and will be briefly discussed at this point.

Evidence of significant heteroscedasticity in comparing the HET-GFF and GFF models (from Table 1) is consistent with findings of Bey and Pinches (1980), suggesting that previous findings of heteroscedasticity are not merely a result of functional form misspecification. Significance tests on the specific variance estimates provide some indication of the underlying heteroscedastic form.<sup>4</sup> For the securities where b or c was significant, c was singly significant in 53 percent of the cases, b was singly significant in 26 percent of the cases, with both b and c significant in the remaining 21 percent of the cases. This finding indicates that the variance expressed as a function of  $R_{mt}^2$  is a predominant form; however, other forms cannot be completely excluded.

---

<sup>4</sup>Of interest are b and c in the variance specification of equation (11). Due to differences in the power of the tests, the significance of the individual coefficients will not always occur when the likelihood ratio test indicates a significant difference.

### C. Market Model Specification and the Estimation of Beta

The series of models tested in this study allows the sensitivity of beta in relation to specification errors to be examined. The mean absolute deviation between the betas of the various models was less than eight percent for all pairwise comparisons. The Spearman rank-order correlation coefficients between the betas for each specification are presented in Table 2. As is apparent from the consistently high correlations between the models, the specification of the market model and related econometric irregularities do not appear to have any notable impact on the estimation of beta.

---

Insert Table 2 about here

---

### D. Nonnormalities and Market Model Specification

To monitor the effects of the various market model specifications on the distributional properties of the underlying process, a measure of skewness and kurtosis was calculated from the market model residuals of each case. Based on the results of Fama and Roll (1971), the Studentized Range was adopted as a test for the normality of the residuals. The average value for these statistics over the 1,042 cases and the percentage of cases where the statistics are significant appears in Table 3.

---

Insert Table 3 about here

---

The presence of nonnormalities in the linear specification is apparent, with significant skewness, kurtosis, and nonnormalacy occurring in at least 50 percent of the securities tested. Consistent with previous studies, the

Table 2  
Effect of specification  
on estimates of beta<sup>a</sup>

	LIN	LOG	GFF
LOG	.995	---	
GFF	.972	.979	---
HET-GFF	.935	.944	.969

<sup>a</sup>Spearman rank-order correlation coefficients between the betas of the four models.



Table 3

Skewness, kurtosis, and studentized range  
for the four empirical models<sup>a</sup>

Model	<u>Skewness</u>		<u>Kurtosis</u>		<u>Studentized Range</u>	
	Mean (Std Dev)	Percent Significant	Mean (Std Dev)	Percent Significant	Mean (Std Dev)	Percent Significant
LIN	.599 (.631)	52.0%	2.276 (2.919)	57.8%	5.891 (.873)	50.6%
LOG	.191 (.532)	28.1%	1.491 (1.995)	47.6%	5.730 (.819)	42.3%
GFF	-.030 (.266)	5.3%	.802 (1.111)	28.9%	5.498 (.739)	30.8%
HET-GFF	-.000 (.124)	0.3%	.375 (.781)	13.8%	5.271 (.635)	18.8%

<sup>a</sup>The three statistics--skewness, kurtosis, and studentized range--are estimated from the residuals of each specification for each security. The mean and standard deviation across the 1,042 securities for each statistic is reported in the table along with the percent of cases where each statistic was significant at the .05 level.

Calculations of skewness and kurtosis are based on Fisher's k-statistics where  $g_1$  (skewness) and  $g_2$  (kurtosis) is estimated for the standardized residuals as:

$$g_1 = \frac{N}{(N-1)(N-2)} \sum z_i^3 \quad g_2 = \frac{N(N+1)}{(N-1)(N-2)(N-3)} \sum z_i^4 - \frac{3(N-1)(N-1)}{(N-2)(N-3)}$$

$$V(g_1) \approx 6/N \quad V(g_2) \approx 24/N$$

The studentized range, as detailed by Fama (1976), is given by

$$SR = \max(z_i) - \min(z_i).$$

results indicate that the log specification reduces these abnormalities to some degree; however it does not completely resolve the problem.

The results of the GFF specification, where the proportion of both significant skewness and kurtosis have been reduced, confirm the impact of the transformations in adjusting for skewness and unequal variances (if, as previously discussed, kurtosis is assumed to be partially attributable to heteroscedasticity). Since the effects of the transformation are confounded in the GFF specification, the results of the more general HET-GFF model are of primary interest.

For the HET-GFF specification, where the effects of the functional form transformations should be concentrated on correcting for nonlinearities and nonnormalities, the presence of skewness has been virtually eliminated. (Further discussion of the transformation appears in the next section.)

The substantial reduction in kurtosis attributable to the heteroscedastic specification provides strong support for the subordinated normal hypothesis. The results indicate that much of the peakedness observed in the return generating process could be attributable to unequal variances and is not necessarily the result of a nonnormal stochastic process. These results are consistent with the findings of Rosenberg and Marathe (1979).

Given that the final specification did not completely remedy the nonnormalities, a more general statistical distribution could still be argued. The rather substantial impact of the general model in reducing kurtosis, however, would suggest that the remaining abnormalities might be resolved by a more exhaustive specification of the heteroscedastic process. Thus, given the promising results of this initial attempt at specifying the variance structure, a more appropriate specification could completely resolve the leptokurtotic issue.

### E. Isolating the Effects of the Transformation Parameter

The series of models tested in this study allows the various remedial effects of the functional form parameters to be partitioned to some degree. The mean value for the estimates of  $\lambda_1$  and  $\lambda_2$  across all securities, along with the percent of cases where the transformations were statistically different from zero and one, are presented in Table 4. The mean differences between the estimates of the GFF and HET-GFF models indicate that the transformation parameter is affected by unequal variances, as noted by Zarembka, and this bias appears to be negative. The frequency of significant transformations is reduced substantially when the GFF model is corrected for unequal variances. From the distribution of the estimated  $\lambda$ 's, there does not seem to be any apparent tendency for the transformations to center around one particular value.

---

Insert Table 4 about here

---

Having isolated the effect of unequal variances on the transformation parameter, the effects of nonnormalities and nonlinearities are still confounded. Unfortunately there is no exact means of disaggregating these effects. A series of tests, however, can be applied to provide evidence of the causes of the significant transformations.

First note, as previously mentioned, that the transformation of the independent variable is conditioning the value of the dependent variable and therefore concentrates only on additivity of effect. The relatively few cases of significant transformations on  $R_{mt}$  reported in Table 4 provide evidence supporting the linearity hypothesis.

The substantial reduction of skewness attributable to the general specification suggests a source of the significance of  $\lambda_1$ . In fact, if all

Table 4  
Estimated transformation parameters

Model	<u><math>\lambda_1</math></u>			<u><math>\lambda_2</math></u>		
	Mean (Std Dev)	$H_0:\lambda=0$ Percent Significant	$H_0:\lambda=1$ Percent Significant	Mean (Std Dev)	$H_0:\lambda=0$ Percent Significant	$H_0:\lambda=1$ Percent Significant
GFF	- .764 (1.239)	27.9%	58.3%	-2.004 (5.955)	11.8%	15.2%
HET-GFF	- .684 (1.535)	19.4%	40.8%	-2.043 (6.412)	7.0%	8.0%

securities with significant skewness in the market model residuals are eliminated from the sample, the occurrence of significant transformations is no more than expected by chance.

The presence of skewness in the residuals could be a result of the distribution of  $R_t$  or misspecification. Thus the nonlinearity-nonnormality issues are still confounded. Further evidence of the source of this abnormality can be provided by examining the distribution of  $R_t$  vis-à-vis  $\epsilon_t$ . If  $R_t$  exhibits patterns of significant skewness,  $\epsilon_t$  would be expected to have a similar distribution (however this still does not conclusively indicate that the irregularities in the error term are exclusively attributable to the distribution of  $R_t$ ). For securities in the linear specification where  $\epsilon_t$  exhibited significant skewness,  $R_t$  was also significantly skewed in 92 percent of the cases.

Although this series of results cannot provide an exact test that isolates the remaining remedial effects, the accumulation of evidence indicates the transformation is concentrating on reducing skewness in the dependent variable. In the absence of skewness, the transformation parameters are no longer significant (for either null hypothesis of  $\lambda=0$  or  $\lambda=1$ ). After adjusting for heteroscedasticity and nonnormalities, the model appears to be insensitive to the functional specification. Therefore, the hypothesis of linearity common to the CAPM and APT cannot be rejected. The "nonlinearities" identified in previous studies using the GFF methodology (e.g., Lee 1976b), can be attributed to distributional aberrations and not to the functional relationship between the market model variables. For market model applications, the estimated transformation parameter appears to be biased by both skewness and heteroscedasticity.

## F. Discussion

From the empirical results of the various market model specifications, it appears that the more general econometric forms are statistically significant in a substantial number of cases. Consistent with the findings of Bey and Pinches (1980), significant heteroscedasticity was found for approximately 39 percent of the securities.

Although the properties of the market model examined in this study have important implications for the underlying theory, in many cases estimation of the market model is simply the first step in a series of tests. In these applications, obtaining an estimate of beta is the predominant purpose for employing the model. The results of this study indicate that beta is surprisingly robust with respect to misspecification of the empirical model. If the researcher is only interested in obtaining an estimate of beta, then the selection of an appropriate statistical specification--linear, logarithmic, or more complex form--does not appear to be a crucial issue.

The more general empirical specifications were successful in substantially reducing the presence of nonnormalities in the market model residuals. The presence of significant skewness was well-documented in the empirical results; however the functional form transformations provide the ability to all but eliminate skewness in the estimation process. The evidence that skewness has a significant impact in estimating the market model has important implications for asset pricing theory. Lee and Wu (1983) have shown that the functional form specification and the quadratic characteristic line of the three-moment CAPM (see Kraus and Litzenberger 1976), are essentially the same with the exception of the assumed investment horizon.<sup>5</sup>

---

<sup>5</sup>Lee and Wu (1983) examine the impact of skewness and kurtosis on asset pricing in a number of contexts. Their findings provide additional insight into the implications of the observed abnormalities on asset pricing.

The ability of the heteroscedastic model to substantially reduce the level of kurtosis provides additional evidence for the subordinated normal hypothesis. The magnitude of effect the HET-GFF model exhibited with respect to kurtosis suggests that the failure to completely eliminate the problem is most likely attributable to the inability to perfectly identify the variance structure.

Estimates of the transformation parameters were negatively biased for this sample in the presence of heteroscedasticity. After adjusting for heteroscedasticity, the transformation parameter associated with the independent variable was significant in only a nominal number of cases. A series of results provided evidence that significant transformations were associated with skewness in the distribution of  $R_t$ . The pattern of results indicate that true nonlinearities could not be detected. Thus the linearity hypothesis for the market model is supported. The findings also indicate that if there is a choice between the linear and logarithmic specifications, the log model appears to be a better surrogate for the more general specifications.

## V. CONCLUSION

Using a sample of 1,042 securities, this study examined the theoretical and statistical implications of certain econometric phenomena occurring in the estimation of the market model. The empirical results have significant implications for the asset pricing theories underlying the single-factor market model and for the estimation of the market model. In summary, the empirical results from the sample tested in this study suggest that:

1. The generalized market models are statistically significant in a substantial number of cases. Significant functional form transformations and heteroscedasticity were found in more than half the securities tested.

2. The effect of the observed econometric aberrations in estimating beta is inconsequential. The estimates of beta are surprisingly robust with respect to misspecification.
3. Approximately 50 percent of the residuals from estimates of the linear market model exhibit significant nonnormalities. Correcting for unequal variances in the heteroscedastic specification reduces the level of kurtosis substantially, thus providing additional support for the subordinated normal hypothesis.
4. Estimates of the GFF transformation parameters in market model applications are negatively biased in the presence of heteroscedasticity. The presence of skewness can be virtually eliminated in the GFF specification. The transformation process appears to concentrate on correcting the dependent variable for nonnormalities in this application. From the results of Lee and Wu (1983), the empirical findings of this paper have important implications for asset pricing theory. Most notably, the results provide initial support for the extension of the CAPM to the three-moment model of Kraus and Litzenberg (1976).
5. The significance of the functional form transformation in this sample is primarily attributable to skewness in the market model residuals. Subsequent tests indicate that the pattern of skewness in the market model residuals is a result of the distribution of  $R_t$  and not the model specification. After removing the effects of skewness and heteroscedasticity on the transformation parameter, the functional form transformations are rarely significant. In this case, the linear and log-linear specifications can not be rejected.
6. Given the choice between a linear and logarithmic specification, the log model appears to reduce some of the statistical irregularities. The log model does not, however, completely resolve these problems.

The single-factor market model will undoubtedly continue to receive considerable application in Financial research. The findings of this study indicate that the parameters of the model are robust with respect to statistical misspecification. The presence of these aberrations though, pose a challenge to the exhaustiveness of the current asset pricing theories.




## REFERENCES

- K. J. Arrow. "The Role of Securities in the Optimal Allocation of Risk-Bearing." Review of Economic Studies 31 (1964), 91-96.
- R. P. Bey and G. E. Pinches. "Additional Evidence of Heteroscedasticity in the Market Model." Journal of Financial and Quantitative Analysis 15 (1980), 299-322.
- G. E. P. Box and D. R. Cox. "An Analysis of Transformations." Journal of the Royal Statistical Society B 26 (1964), 211-243.
- Hui-shyong Chang. "Functional Forms and the Demand for Meat in the United States." Review of Economics and Statistics 59 (1977), 355-359.
- P. K. Clark. "A Subordinated Stochastic Process Model with Finite Variance for Speculative Prices." Econometrica 41 (1973), 135-159.
- G. Debreu. The Theory of Value. New York: Wiley, 1959.
- N. R. Draper and D. R. Cox. "On Distributions and Their Transformation to Normality." Journal of the Royal Statistical Society B 31 (1969), 472-476.
- F. Fabozzi, J. Francis and C. Lee. "Generalized Functional Form for Mutual Fund Returns." Journal of Financial and Quantitative Analysis 15 (1980), 1107-1119.
- E. Fama. "The Behavior of Stock Market Prices." Journal of Business 38 (1965), 34-105.
- E. Fama and R. Roll. "Parameter Estimates for Symmetric Stable Distributions." Journal of the American Statistical Association 66 (1971), 331-338.
- S. Goldfeld and R. E. Quandt. Nonlinear Methods in Econometrics. Amsterdam: North-Holland, 1972.
- A. Kraus and R. Litzenberger. "Skewness Preference and the Valuation of Risk Assets." Journal of Finance 31 (1976), 1085-1100.
- K. Lahairi and D. Egy. "Joint Estimation and Testing for Functional Form and Heteroskedasticity." Journal of Econometrics 15 (1981), 299-307.
- C. F. Lee. "Functional Form and the Dividend Effect in the Electric Utility Industry." Journal of Finance 31 (1976a), 1481-1486.
- C. F. Lee. "Investment Horizon and the Functional Form of the Capital Asset Pricing Model." Review of Economics and Statistics 58 (1976b), 356-363.
- C. F. Lee. "Functional Form, Skewness Effect and the Risk-Return Relationship." Journal of Financial and Quantitative Analysis 12 (1977), 55-72.

- C. F. Lee and C. Wu. "The Impacts of Skewness and Kurtosis on the Risk Estimation and Determination." Working Paper, University of Illinois (1983).
- J. Lintner. "The Lognormality of Portfolio Returns, Portfolio Selection and Market Equilibrium." Unpublished manuscript (1975).
- B. Mandelbrot. "The Variation of Certain Speculative Prices." Journal of Business 36 (1963), 394-419.
- B. McDonald and M. Morris. "The Existence of Heteroscedasticity and Its Effects on Estimates of the Market Model Parameters." The Journal of Financial Research 6 (1983), 115-126.
- J. B. Ramsey and P. Zarembka. "Specification Error Tests and Alternative Functional Forms of the Aggregate Production Function." Journal of the American Statistical Association 66 (1971), 471-477.
- B. Rosenberg and V. Marathe. "Tests of Capital Asset Pricing Hypotheses." 115-223 in H. Levy, ed., Research in Finance. Connecticut: JAI Press, 1979.
- S. Ross. "The Arbitrage Theory of Capital Asset Pricing." Journal of Economic Theory 13 (1976), 341-360.
- J. Spitzer. "The Demand for Money, the Liquidity Trap, and Functional Forms." International Economic Review 17 (1976), 220-227.
- J. Spitzer. "A Monte Carlo Investigation of the Box-Cox Transformation in Small Samples." Journal of the American Statistical Association 73 (1978), 488-495.
- S. Sunder. "Stationarity of Market Risk: Random Coefficients Tests for Individual Stocks." Journal of Finance 35 (1980), 883-896.
- R. Westerfield. "The Distribution of Common Stock Price Changes: An Application of Transactions Time and Subordinated Stochastic Models." Journal of Financial and Quantitative Analysis 12 (1977), 743-765.
- P. Zarembka. "Transformation of Variables in Econometrics." 81-104 in Paul Zarembka, ed., Frontiers in Econometrics. New York: Academic Press, 1974.





ECKMAN  
INDERY INC.   
JUN 95  
ind - To - Please N MANCHESTER.  
INDIANA 46962

UNIVERSITY OF ILLINOIS URBANA



3 0112 060296123